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On Mutually Nearest and Mutually Furthest Points in Reflexive Banach Spaces[1]

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Let G be a nonempty closed (resp. bounded closed) subset in a reflexive strictly convex Kadec Banach space X. Let $\mathscr{K}(X)$ denote the space of all nonempty compact convex subsets of X endowed with the Hausdorff distance. Moreover, let $\mathscr{K}_G(X)$ denote the closure of the set $\{A \in \mathscr{K}(X) : A \cap G = \emptyset\}$. A minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is said to be well posed if it has a unique solution (x_0, z_0) and every minimizing (resp. maximizing) sequence converges strongly to (x_0, z_0). We prove that the set of all $A \in \mathscr{K}_G(X)$ (resp. $A \in \mathscr{K}(X)$) such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed contains a dense G_δ-subset of $\mathscr{K}_G(X)$ (resp. $\mathscr{K}(X)$), extending the results in uniformly convex Banach spaces due to Blasi, Myjak and Papini.

1. INTRODUCTION

Let X be a real Banach space. We denote by $\mathscr{B}(X)$ the space of all nonempty closed bounded subsets of X. For a closed subset G of X and $A \in \mathscr{B}(X)$, we set

$$\lambda_{AG} = \inf\{\|z - x\| : x \in A, z \in G\},$$

and for $G \in \mathscr{B}(X)$, we set

$$\mu_{AG} = \sup\{\|z - x\| : x \in A, z \in G\}.$$

Given a nonempty closed subset G of X (resp. $G \in \mathscr{B}(X)$), according to [9], a pair (x_0, z_0) with $x_0 \in A$, $z_0 \in G$ is called a solution of the minimization (resp. maximization) problem, denoted by $\min(A, G)$ (resp. $\max(A, G)$), if $\|x_0 - z_0\| = \lambda_{AG}$ (resp. $\|x_0 - z_0\| = \mu_{AG}$). Moreover, any sequence $\{(x_n, z_n)\}$,

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$x_n \in A$, $z_n \in G$, such that $\lim_{n\to\infty} \|x_n - z_n\| = \lambda_{AG}$ (resp. $\lim_{n\to\infty} \|x_n - z_n\| = \mu_{AG}$) is called a minimizing (resp. maximizing) sequence. A minimization (resp. maximization) problem is said to be well posed if it has a unique solution (x_0, z_0), and every minimizing (resp. maximizing) sequence converges strongly to (x_0, z_0).

Set

$$\mathscr{C}(X) = \{A \in \mathscr{B}(X) : A \text{ is convex}\},$$

and let $\mathscr{C}(X)$ be endowed with the Hausdorff distance h defined as follows:

$$h(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|\}, \qquad \forall A, B \in \mathscr{C}(X).$$

As is well known, under such metric, $\mathscr{C}(X)$ is complete.

In [9], the authors considered the well posedness of the minimization and maximization problems. If X is a uniformly convex Banach space they proved that the set of all $A \in \mathscr{C}_G(X)$ (resp. $A \in \mathscr{C}(X)$), such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed, is a dense G_δ-subset of $\mathscr{C}_G(X)$ (resp. $\mathscr{C}(X)$), where $\mathscr{C}_G(X)$ is the closure of the set $\{A \in \mathscr{C}(X) : \lambda_{AG} > 0\}$.

Furthermore, let

$$\mathscr{K}(X) = \{A \in \mathscr{C}(X) : A \text{ is compact}\}$$

and $\mathscr{K}_G(X) = \mathscr{K}(X) \cap \mathscr{C}_G(X)$. Clearly, X can be embedded as a subset of $\mathscr{K}(X)$ in a natural way that, for any $x \in X$, $A_x \in \mathscr{K}(X)$ is defined by $A_x = \{x\}$.

It is our purpose in the present note to extend the results, with a completely different approach, to a reflexive strictly convex Kadec Banach space. We prove that if X is a reflexive strictly convex Kadec Banach space, then the set of all $A \in \mathscr{K}_G(X)$ (resp. $A \in \mathscr{K}(X)$), such that the minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is well posed, contains a dense G_δ-subset of $\mathscr{K}_G(X)$ (resp. $\mathscr{K}(X)$).

It should be noted that the problems considered here are in the spirit of Stechkin [27]. Some further developments of Stechkin's ideas can be founded in [2–6, 8, 11–17, 20, 24, 26] and in the monograph [10], while some generic results in spaces of convex sets and bounded sets can be founded in [2, 3, 7, 19, 21].

In sequel, let X^* denote the dual of X. We use $B(x, r)$ to denote the closed ball with center at x and radius r. As usual, if $A \subset X$, by $\bar{A}$ and diam A we mean the closure and the diameter of A, respectively, while $\overline{\text{co}}\, A$ stands for the closed convex hull of A.

DEFINITION 1.1. Let D be an open subset of X. A real-valued function f on D is said to be Frechet differentiable at $x \in D$ if there exists an $x^* \in X^*$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$

x^* is called the Frechet differential at x which is denoted by $Df(x)$.

The following proposition on the Frechet differentiability of Lipschitz functions due to [24] is useful.

PROPOSITION 1.1. *Let f be a locally Lipschitz continuous function on an open set D of a Banach space with equivalent Frechet differentiable norm* (*in particular, X reflexive will do*). *Then f is Frechet differentiable on a dense subset of D.*

DEFINITION 1.2. A Banach space X is said to be (sequentially) Kadec provided that for each sequence $\{x_n\} \subset X$ which converges weakly to x with $\lim_{n \to \infty} \|x_n\| = \|x\|$ we have $\lim_{n \to \infty} \|x_n - x\| = 0$.

DEFINITION 1.3. A Banach space X is said to be strongly convex provided it is reflexive, Kadec and strictly convex.

We also need a result concerning the characterization of strongly convex spaces, which is due to Konjagin [15], see also Borwein and Fitzpatrick [5].

PROPOSITION 1.2. *A Banach space X is strongly convex if and only if for every closed nonempty subset G of X there is a dense set of points $X \setminus G$ possessing unique nearest points.*

2. MINIMIZATION PROBLEMS

Let $x \in X$, $A \in \mathscr{K}(X)$ and G be a closed subset of X. We set

$$d_G(x) = \inf_{z \in G} \|x - z\|,$$

$$d_G(A) = \inf_{x \in A} d_G(x) = \lambda_{AG}$$

and

$$P_A(G) = \{x \in A : d_G(x) = d_G(A)\}.$$

Then

$$|d_G(A) - d_G(B)| \leqslant h(A, B), \qquad \forall A, B \in \mathscr{K}(X).$$

For $A \in \mathscr{K}(X)$, let f_A be the functional on X defined as follows:

$$f_A(x) = d_G(A + x), \qquad \forall x \in X.$$

Then f_A is 1-Lipschitz and satisfies $f_A(x) = f_{A+x}(0)$.

LEMMA 2.1. *Suppose that* f_A *is Frechet differentiable at* $x = 0$ *with* $Df_A(0) = x^*$. *Then* $\|x^*\| = 1$ *and for any* $x_0 \in P_A(G)$, $\{z_n\} \subset G$ *with* $\lim_{n\to\infty} \|x_0 - z_n\| = d_G(x_0)$, *we have*

$$d_G(x_0) = \lim_{n\to\infty} \langle x^*, x_0 - z_n \rangle.$$

Proof. Let x_0, $\{z_n\}$ satisfy the assumptions of the lemma. Then for each $1 \geqslant t > 0$,

$$\begin{aligned} f_A(t(z_n - x_0)) - f_A(0) &= d_G(A + t(z_n - x_0)) - d_G(A) \\ &\leqslant \|x_0 + t(z_n - x_0) - z_n\| - d_G(A) \\ &= (1-t)\ \|x_0 - z_n\| - d_G(A) \\ &= -t\ \|x_0 - z_n\| + [\ \|x_0 - z_n\| - d_G(A)]. \end{aligned}$$

Let $t_n = 2^{-n} + [\ \|x_0 - z_n\| - d_G(A)]^{1/2}$. Then from the Frechet differentiability of $f_A(x)$ at $x = 0$, we have that

$$\lim_{n\to\infty} \left[\frac{f_A(t_n(z_n - x_0)) - f_A(0)}{t_n} - \langle x^*, z_n - x_0 \rangle \right] = 0,$$

so that

$$\liminf_{n\to\infty} [-\|x_0 - z_n\| + \langle x^*, x_0 - z_n \rangle] \geqslant 0$$

and

$$d_G(A) = \lim_{n\to\infty} \|x_0 - z_n\| \leqslant \liminf_{n\to\infty} \langle x^*, x_0 - z_n \rangle.$$

Note that $\|x^*\| \leqslant 1$ since f_A is 1-Lipschitz. It follows that

$$\lim_{n\to\infty} \|x_0 - z_n\| \geqslant \lim_{n\to\infty} \|x^*\|\ \|x_0 - z_n\| \geqslant \limsup_{n\to\infty} \langle x^*, x_0 - z_n \rangle.$$

Comparison of the last two inequalities shows the desired results, proving the lemma.

LEMMA 2.2. *The set-valued map $P_A(G)$ with respect to A is upper semi-continuous in the sense that for each $A_0 \in \mathcal{K}_G(X)$ and any open set U with $P_{A_0}(G) \subset U$, there exists $\delta > 0$ such that for any $A \in \mathcal{K}_G(X)$ with $h(A, A_0) < \delta$, $P_A(G) \subset U$.*

Proof. Suppose on the contrary that there exist $\{A_n\} \subset \mathcal{K}_G(X)$ and $A \in \mathcal{K}_G(X)$ with $\lim_{n \to \infty} h(A_n, A) = 0$, such that $P_{A_n}(G) \not\subset U$ for some open subset U with $P_A(G) \subset U$ and each n. Let $x_n \in P_{A_n}(G) \backslash U$ for any n. Note that $\bigcup_n A_n$ is relatively compact and $\{x_n\} \subset \bigcup_n A_n$. It follows that there exists a subsequence, denoted by itself, such that $\lim_{n \to \infty} \|x_n - x_0\| = 0$ for some $x_0 \in X$. Clearly, $x_0 \notin U$. However, by $\lim_{n \to \infty} h(A_n, A) = 0$, there exists $\{a_n\} \subset A$ such that $\lim_{n \to \infty} \|x_n - a_n\| = 0$ so that

$$\limsup_{n \to \infty} \|a_n - x_0\| \leqslant \lim_{n \to \infty} \|x_n - a_n\| + \lim_{n \to \infty} \|x_n - x_0\| = 0$$

and $x_0 \in A$. Furthermore, for each n,

$$\begin{aligned} \inf_{z \in G} \|z - x_0\| &\leqslant \inf_{z \in G} \|z - x_n\| + \|x_n - x_0\| \\ &\leqslant d_G(A) + h(A_n, A) + \|x_n - x_0\|. \end{aligned}$$

which shows that $x_0 \in P_A(G)$, contradicting that $x_0 \notin U$. The proof is complete.

Let

$$L_n(G) = \left\{ A \in \mathcal{K}_G(X) : \begin{array}{c} \inf\{\langle x^*, x - z \rangle : z \in G \cap B(x, d_G(x) + \delta), \\ x \in P_A(G)\} > (1 - 2^{-n})\, d_G(A), \\ \text{for some } \delta > 0,\ x^* \in X^* \text{ with } \|x^*\| = 1. \end{array} \right\}$$

Also let

$$L(G) = \bigcap_n L_n(G).$$

LEMMA 2.3. *Suppose that X is reflexive. Then $L(G)$ is a dense G_δ-subset of $\mathcal{K}_G(X)$.*

Proof. To show that $L(G)$ is a G_δ-subset of $\mathscr{K}_G(X)$, we only need prove that $L_n(G)$ is open for each n. Let $A \in L_n(G)$. Then there exist $x^* \in X^*$ with $\|x^*\| = 1$ and $\delta > 0$ such that

$$\beta = \inf\{\langle x^*, x - z\rangle : x \in P_A(G),\, z \in G \cap B(x, d_G(x) + \delta)\} \\ -(1 - 2^{-n})\, d_G(A) > 0.$$

Let $\lambda > 0$ be such that $\lambda < \min\{(\delta/2), (\beta/2)\}$. It follows from Lemma 2.2 that there exists $0 < \varepsilon < \lambda$ such that for any $F \in \mathscr{K}_G(X)$ with $h(F, A) < \varepsilon$ and each $y \in P_F(G)$ there exists $x \in P_A(G)$ satisfying $\|y - x\| < \lambda$. For $\delta^* = \delta - 2\lambda$ we have

$$H = G \cap B(y, d_G(y) + \delta^*) \subset G \cap B(x, d_G(x) + \delta).$$

Thus if $z \in H$,

$$\langle x^*, x - z\rangle \geqslant \beta + (1 - 2^{-n})\, d_G(A)$$

and

$$\langle x^*, y - z\rangle > \beta + (1 - 2^{-n})\, d_G(F) - \lambda.$$

Then

$$\inf\{\langle x^*, y - z\rangle : z \in H,\, y \in P_F(G)\} > (1 - 2^{-n})\, d_G(F)$$

and $F \in L_n(G)$ for all $F \in \mathscr{K}_G(X)$ with $h(F, A) < \varepsilon$, which implies that $L_n(G)$ is open in $\mathscr{K}_G(X)$.

In order to prove the density of $L(G)$ in $\mathscr{K}_G(X)$, from Proposition 1.1, it suffices to prove that if $f_A(x)$ is Frechet differentiable at $x = 0$ then $A \in L(G)$.

Suppose on the contrary that for some n there exist $\{x_m\} \subset P_A(G)$ and $\{z_m\} \subset G \cap B(x_m, d_G(x_m) + 2^{-m})$ such that

$$\langle x^*, x_m - z_m\rangle \leqslant (1 - 2^{-n})\, d_G(A), \qquad \forall m,$$

where $x^* = Df_A(0)$. With no loss of generality, we assume that $\lim_{m \to \infty} \|x_m - x_0\| = 0$ for some $x_0 \in P_A(G)$. Observe that $\lim_{m \to \infty} \|x_m - z_m\| = d_G(A)$. Then $\lim_{m \to \infty} \|x_0 - z_m\| = d_G(A)$. Thus Lemma 2.1 implies that

$$\lim_{m \to \infty} \langle x^*, x_0 - z_m\rangle = d_G(A)$$

so that

$$\lim_{m \to \infty} \langle x^*, x_m - z_m\rangle = d_G(A)$$

which contradicts that

$$\langle x^*, x_m - z_m \rangle \leqslant (1-2^{-n})\, d_G(A), \qquad \forall m.$$

This completes the proof.

LEMMA 2.4. *Suppose X is a reflexive Kadec Banach space. Let $A \in L(G)$. Then any minimizing sequence $\{(x_n, z_n)\}$ with $x_n \in A$, $z_n \in G$ has a subsequence which converges strongly to a solution of the minimization problem* $\min(A, G)$.

Proof. Let $A \in L(G)$. Then $A \in L_m(G)$ for any $m = 1, 2, \ldots$. By the definition of $L_m(G)$, there exist $\delta_m > 0$, $x_m^* \in X^*$, $\|x_m^*\| = 1$ such that

$$\inf\{\langle x_m^*, x - z\rangle : z \in G \cap B(x, d_G(x) + \delta_m), x \in P_A(G)\} > (1 - 2^{-m})\, d_G(A).$$

Let $\{(x_n, z_n)\}$ with $x_n \in A$, $z_n \in G$ be any minimizing sequence. With no loss of generality, we assume that $x_n \to x_0$ strongly and $z_n \to z_0$ weakly as $n \to \infty$ for some $x_0 \in P_A(G)$, $z_0 \in X$, since A is compact and X is reflexive. Then we have that

$$\|x_0 - z_0\| \leqslant \liminf_{n\to\infty} \|x_0 - z_n\| = d_G(A).$$

We also assume that $\delta_n \leqslant \delta_m$ if $m < n$ and $z_n \in G \cap B(x_0, d_G(x_0) + \delta_m)$ for all $n > m$. Thus,

$$\langle x_m^*, x_0 - z_n\rangle > (1 - 2^{-m})\, d_G(A), \qquad \forall n > m$$

and

$$\langle x_m^*, x_0 - z_0\rangle > (1 - 2^{-m})\, d_G(A), \qquad \forall m.$$

Hence we have

$$\|x_0 - z_0\| \geqslant \limsup_{m\to\infty} \langle x_m^*, x_0 - z_0\rangle \geqslant d_G(A).$$

This shows that $\|x_0 - z_0\| = d_G(A)$. Now the fact that X is Kadec implies that $\lim_{n\to\infty} \|z_n - z_0\| = 0$ and $z_0 \in G$. Clearly, (x_0, z_0) is a solution of the minimization problem $\min(A, G)$ and completes the proof.

Let

$$Q_n(G) = \left\{A \in \mathscr{K}_G(X) : \operatorname{diam} P_A(G) < \frac{1}{n}\right\}$$

and let

$$Q(G) = \bigcap_n Q_n(G).$$

LEMMA 2.5. *Suppose that X is reflexive Kadec Banach space. Then $Q(G)$ is a dense G_δ-subset of $\mathcal{K}_G(X)$.*

Proof. Given n and $A \in Q_n(X)$, we define

$$c = \frac{1}{n} - \operatorname{diam} P_A(G)$$

and

$$U = \left\{ x \in X : d_{P_A(G)}(x) < \frac{c}{3} \right\}.$$

Then

$$\operatorname{diam} U < \operatorname{diam} P_A(G) + \frac{2c}{3} < \frac{1}{n}.$$

It follows from Lemma 2.2 that there exists $\lambda > 0$ such that $P_F(G) \subset U$ for any $F \in \mathcal{K}(X)$ with $h(F, A) < \lambda$. This shows $\operatorname{diam} P_F(G) < (1/n)$ for any $F \in \mathcal{K}(X)$ with $h(F, A) < \lambda$ so that $Q_n(G)$ is open and $Q(G)$ is a G_δ-subset of $\mathcal{K}_G(X)$.

Now let us prove that $Q(G)$ is dense. From Lemma 2.3 and 2.4 it suffices to prove that for any $A \in L(G)$ and a solution (x_0, z_0) of $\min(A, G)$, the set A_α defined by

$$A_\alpha = \overline{co}\,(A \cup \{x_\alpha\})$$

is in $Q(G)$ for all $0 < \alpha < 1$, where $x_\alpha = \alpha x_0 + (1 - \alpha)\, z_0$.

Observe that for each $0 < \alpha < 1$, if $x \in A_\alpha$, $x \neq x_\alpha$, then $x = ta + (1-t)\, x_\alpha$ for some $0 < t \leqslant 1$ and $a \in A$. Set $a_0 = ta + (1-t)\, x_0$. Then $a_0 \in A$ and

$$\begin{aligned}
\inf_{z \in G} \|z - x\| &\geqslant \inf_{z \in G} \|z - a_0\| - \|a_0 - x\| \\
&\geqslant \|z_0 - x_0\| - (1-t)\, \|x_0 - x_\alpha\| \\
&= (1 - (1-t)(1-\alpha))\, \|z_0 - x_0\| \\
&> \alpha\, \|z_0 - x_0\| = \|z_0 - x_\alpha\| \geqslant \lambda_{A_\alpha G}.
\end{aligned}$$

This shows $P_{A_\alpha}(G) = x_\alpha$ and proves the lemma.

Now we are ready to give the main theorem of this section.

THEOREM 2.1. *Suppose that X is a strongly convex Banach space. Let G be a closed subset of X. Then the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem* $\min(A, G)$ *is well posed contains a dense G_δ-subset of $\mathcal{K}_G(X)$.*

Proof. It suffices to prove that $\min(A, G)$ is well posed if $A \in Q(G) \cap L(G)$, as $Q(G) \cap L(G)$ is a dense G_δ-subset of $\mathcal{K}_G(X)$.

We first show that $\min(A, G)$ has a unique solution. Suppose there is $A \in Q(G) \cap L(G)$ such that $\min(A, G)$ has two solutions (x_0, z_0), (x_1, z_1). Clearly $x_1 = x_0$ because $A \in Q(G)$. On the other hand, since $A \in L(G)$, for each n, there exists $x_n^* \in X$, $\|x_n^*\| = 1$ satisfying

$$\langle x_n^*, x_0 - z_i \rangle > (1 - 2^{-n})\, d_G(A), \qquad i = 0, 1$$

so that

$$\|x_0 - z_0 + x_0 - z_1\| \geqslant \limsup_{n \to \infty} \langle x_n^*, x_0 - z_0 + x_0 - z_1 \rangle \geqslant 2 d_G(A).$$

Thus, using the strict convexity of X, we have $z_0 = z_1$, proving the uniqueness.

Now let (x_n, z_n) with $x_n \in A$, $z_n \in G$ be any minimizing sequence. Then from the uniqueness and Lemma 2.4 it follows that (x_n, z_n) converges strongly to the unique solution of the minimization problem $\min(A, G)$. The proof is complete.

Remark 2.1. Theorem 2.1 is a multivalued version of a theorem due to Lau [17].

Note that if $\min(A, G)$ has a unique solution (x_0, z_0), then x_0 has a unique nearest point in G. This, with Proposition 1.2 and Theorem 2.1, make us prove the following theorem.

THEOREM 2.2. *Let X be a Banach space. Then the following statements are equivalent*:

(i) *X is strongly convex*;

(ii) *for every closed non-empty subset G of X, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem* $\min(A, G)$ *is well posed contains a dense G_δ-subset of $\mathcal{K}_G(X)$*;

(iii) *for every closed non-empty subset G of X, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem* $\min(A, G)$ *is well posed contains a dense subset of $\mathcal{K}_G(X)$.*

Proof. By Theorem 2.1, it suffices to prove that (iii) implies (i). For any fixed $x \in X \backslash G$ and any $\varepsilon > 0$, $\varepsilon < d_G(x)$, let A_ε denote the closed ball at x with radius $\varepsilon/2$. From (iii) it follows that there exists $B_\varepsilon \in \mathscr{K}_G(X)$ such that $h(A_\varepsilon, B_\varepsilon) < (\varepsilon/2)$ and $\min(B_\varepsilon, G)$ is well posed so that $\min(B_\varepsilon, G)$ has a unique solution (x', z'). Thus,

$$\|x' - x\| \leqslant h(A_\varepsilon, B_\varepsilon) + \frac{\varepsilon}{2} < \varepsilon$$

and x' has a unique nearest point z' in G. Using Proposition 1.2, we complete the proof.

Remark 2.2. Let X be a space of finite dimensions. It follows from Remark 3.4 in [9] that Theorem 2.1 and so Theorem 2.2 may not hold if $\mathscr{K}_G(X)$ is replaced by $\mathscr{K}(X)$.

3. MAXIMIZATION PROBLEMS

In order to establish the well posedness result of maximization problems we need some lemmas on furthest points.

Let E be a real Banach space and G be a bounded closed subset of E. We set

$$F_G(x) = \sup_{z \in G} \|x - z\|, \qquad \forall x \in E.$$

Thus $z \in G$ is called a furthest point of x with respect to G if $\|z - x\| = F_G(x)$. The set of all furthest point of x with respect to G is denoted by $R_G(x)$, that is,

$$R_G(x) = \{z \in G : \|z - x\| = F_G(x)\}.$$

LEMMA 3.1. *Suppose that $F_G(\cdot)$ is Frechet differentiable at $x \in E$ with $DF_G(x) = x^*$. Then $\|x^*\| = 1$, and for any $\{z_n\} \subset G$ with $\lim_{n \to \infty} \|x - z_n\| = F_G(x)$, we have*

$$\lim_n \langle x^*, x - z_n \rangle = F_G(x).$$

Proof. Let $\{z_n\} \subset G$ such that $\lim_{n \to \infty} \|x - z_n\| = F_G(x)$. It follows that for $\forall t < 0$,

$$F_G(x + t(z_n - x)) - F_G(x) \geqslant -t\, \|x - z_n\| + \|x - z_n\| - F_G(x).$$

Taking $t_n < 0$, $t_n \to 0$ with $t_n^2 > F_G(x) - \|x - z_n\|$, we have

$$\lim_n \left(\frac{F_G(x + t_n(z_n - x)) - F_G(x)}{t_n} - \langle x^*, z_n - x \rangle \right) = 0.$$

This implies that

$$\liminf_n \left(- \|x - z_n\| - t_n + \langle x^*, x - z_n \rangle \right) \geqslant 0.$$

Now $\|x^*\| \leqslant 1$ since $F_G(\cdot)$ is 1-Lipschitz. It follows that

$$\begin{aligned} F_G(x) &\leqslant \liminf_n \langle x^*, x - z_n \rangle \\ &\leqslant \limsup_n \langle x^*, x - z_n \rangle \\ &\leqslant \lim_n \|x^*\| \, \|x - z_n\| \\ &\leqslant \|x^*\| \, F_G(x) \leqslant F_G(x). \end{aligned}$$

This shows that $\|x^*\| = 1$ and

$$\lim_n \langle x^*, x - z_n \rangle = F_G(x).$$

The proof is complete.

For $y \in E$, define

$$S = \overline{\operatorname{span} G}, \qquad E_y = S \oplus \operatorname{span}\{y\},$$

and let $J(G)$ denote the set of all $y \in E$ such that $F_G(\cdot)$ is Frechet differentiable at y when $F_G(\cdot)$ is restricted on the subspace E_y.

LEMMA 3.2. *$J(G)$ is a G_δ-subset of E.*

Proof. For any $y \in E$, let $J_y(G)$ denote the set of all points $x \in E_y$ such that $F_G(\cdot)$ is Frechet differentiable at x when $F_G(\cdot)$ is restricted on the subspace E_y. Clearly, $J_y(G) \subset J(G)$ for any $y \in E$. Then $J(G) = \bigcup_{y \in E} J_y(G)$ is a G_δ-subset of E from Proposition 1.25 of [23] or [20] since $F_G(\cdot)$ is convex on E.

LEMMA 3.3. *Let $\mathscr{D}$ be a closed convex subset of E. Suppose that S is reflexive and $S \subset \mathscr{D}$. Then $\mathscr{D} \cap J(G)$ is a dense G_δ-subset of $\mathscr{D}$*

Proof. From Lemma 3.2, it suffices to prove that $\mathscr{D} \cap J(G)$ is dense in $\mathscr{D}$. Toward this end, for fixed $y \in \mathscr{D}$, set

$$O = \{\alpha y + x : x \in S, 0 < \alpha < 1\}.$$

Then $O \subset \mathscr{D}$ is open in E_y and E_y is reflexive. It follows from the convexity of the function F_G and Proposition 1.1 (see also [23]) that $F_G(\cdot)$ is Frechet differentiable on a dense subset of E_y when $F_G(\cdot)$ is restricted on the subspace E_y, so that there exists $\{x_n\} \subset O$ such that $F_G(\cdot)$ is Frechet differentiable at x_n and $x_n \to y$. Observe that $E_{x_n} = E_y$ for any n. It follows that $\mathscr{D} \cap J(G)$ is dense in $\mathscr{D}$. The proof is complete.

Now we suppose $\mathscr{K}(X)$ to be endowed with the addition and multiplication as follows:

$$A + B = \{a + b : a \in A, b \in B\}, \qquad \forall A, B \in \mathscr{K}(X),$$

$$\lambda A = \{\lambda a : a \in A\}, \qquad \forall A \in \mathscr{K}(X), \quad \lambda \geqslant 0.$$

Then it follows from the proof of Theorem 2 in [25] that

LEMMA 3.4. *Suppose that X is a reflexive Banach space. Then there exists a Banach space $(E, \|\cdot\|_E)$ such that $\mathscr{K}(X)$ is embedded as a convex cone in such a way that*

(i) *the embedding is isometric, that is, $\forall A, B \in \mathscr{K}(X)$, $h(A, B) = \|A - B\|_E$;*

(ii) *addition in E induces addition in $\mathscr{K}(X)$;*

(iii) *multiplication by nonnegative scalars in E induces the corresponding operation in $\mathscr{K}(X)$;*

(iv) *linear operation in E induces linear operation in X.*

Thus, from $X \subset E$, for $G \in \mathscr{B}(X)$, $A \in \mathscr{K}(X) \subset E$, we have

$$R_G(A) = \{z \in G : \|A - z\|_E = F_G(A)\} = \{z \in G : \sup_{x \in A} \|x - z\| = \mu_{AG}\}.$$

LEMMA 3.5. *Suppose that X is reflexive Kadec Banach space. Let E be given by Lemma 3.4 and $G \in \mathscr{B}(X)$. Then for $A \in J(G)$ any sequence $\{z_n\} \subset G$ with $\lim_{n \to \infty} \sup_{x \in A} \|x - z_n\| = \mu_{AG}$ has a subsequence which converges strongly to an element of $R_G(A)$.*

Proof. Let $A \in J(G)$ and let $\{z_n\} \subset G$ such that $\lim_{n \to \infty} \sup_{x \in A} \|x - z_n\| = \mu_{AG}$. Using Lemma 3.1 and Lemma 3.4, there exists $x_E^* \in E^*$ such that $\|x_E^*\| = 1$ and

$$\lim_n \langle x_E^*, A - z_n \rangle = F_G(A).$$

By the reflexivity of X, there exists a subsequence z_n, denoted by itself, which converges weakly to $z \in X$. Thus,

$$\|A - z\|_E \geqslant \langle x_E^*, A - z \rangle = \lim_n \langle x_E^*, A - z_n \rangle = F_G(A).$$

Note that

$$\|A - z\|_E \leqslant \lim_n \|A - z_n\|_E \leqslant F_G(A).$$

Then

$$\lim_n \|A - z_n\|_E = \|A - z\|_E.$$

Since A is compact, we take $a_0 \in A$ and $x^* \in X^*$, $\|x^*\| \leqslant 1$ such that

$$\|a_0 - z\| = \sup_{a \in A} \|a - z\| = F_G(A)$$

and

$$\langle x^*, a_0 - z \rangle = \|a_0 - z\| = F_G(A).$$

From the fact that $\{x_n\}$ converges weakly to z, we have

$$\begin{aligned} \|a_0 - z\| &= \langle x^*, a_0 - z \rangle = \lim_n \langle x^*, a_0 - z_n \rangle \\ &\leqslant \liminf_n \|a_0 - z_n\| \leqslant \limsup_n \|a_0 - z_n\| \\ &\leqslant \sup_{a \in A, x \in G} \|a - x\| = F_G(A), \end{aligned}$$

so that

$$\lim_n \|a_0 - z_n\| = \|a_0 - z\|.$$

Then the fact that X is Kadec shows $\lim_{n \to \infty} \|z_n - z\| = 0$ and $z \in G$, proving the lemma.

Let

$$V_n = \left\{ A \in \mathscr{K}(X) : \operatorname{diam} R_A(G) < \frac{1}{n} \right\}$$

and let

$$V(G) = \bigcap_n V_n(G),$$

where $R_A(G) = \{x \in A : \sup_{z \in G} \|z - x\| = \mu_{AG}\}$.

LEMMA 3.6. *Suppose that X is reflexive Kadec Banach space. Then $V(G)$ is a dense G_δ-subset of $\mathscr{K}(X)$.*

Proof. Exactly as in the proof of Lemma 2.5 we can obtain that $V(G)$ is a G_δ-subset of $\mathscr{K}(X)$. To prove the density, for any $A \in J(G)$, by Lemma 3.5, we may take (x_0, z_0) to be a solution of $\max(A, G)$ with $x_0 \in A$, $z_0 \in G$, and let $x_\alpha = \alpha x_0 + (1 - \alpha) z_0$ for $\alpha > 1$. We define $A_\alpha = \overline{co}(A \cup \{x_\alpha\})$. Thus, using Lemma 3.3, the proof will be completed if we can prove that $A_\alpha \in V(G)$ for all $\alpha > 1$.

Now for any $x \in A_\alpha$ if $x \neq x_\alpha$ then $x = t x_\alpha + (1 - t) a$ for some $a \in A$ and $0 \leqslant t < 1$. Thus we have

$$\begin{aligned}
\sup_{z \in G} \|z - x\| &\leqslant t \sup_{z \in G} \|z - x_\alpha\| + (1 - t) \sup_{z \in G} \|z - a\| \\
&\leqslant t[\sup_{z \in G} \|z - x_0\| + \|x_0 - x_\alpha\|] + (1 - t) \|z_0 - x_0\| \\
&= t[\|z_0 - x_0\| + (\alpha - 1) \|z_0 - x_0\|] + (1 - t) \|z_0 - x_0\| \\
&= (t\alpha + 1 - t) \|z_0 - x_0\| < \alpha \|z_0 - x_0\| \\
&= \|z_0 - x_\alpha\| \leqslant \mu_{A_\alpha G}.
\end{aligned}$$

This implies that $R_{A_\alpha}(G) = x_\alpha$ and $A_\alpha \in V(G)$ for all $\alpha > 1$.

The main theorem of this section is stated as follows:

THEOREM 3.1. *Suppose that X is a strongly convex Banach space and $G \in \mathscr{B}(X)$. Then the set of all $A \in \mathscr{K}(X)$ such that the maximization problem $\max(A, G)$ is well posed contains a dense G_δ-subset of $\mathscr{K}(X)$.*

Proof. Note that for any $A \in J(G) \cap \mathscr{K}(X)$, $R_G(A) = \{z_0\}$ is a singleton.

In fact, suppose that $R_G(A)$ contains at least two distinct elements $x_0, x_1 \in G$. Then by Lemma 3.1 there exists $x^* \in E^*$ satisfying

$$\langle x^*, A - x_0 \rangle = \langle x^*, A - x_1 \rangle = F_G(A).$$

Hence

$$\|A - x_0 + A - x_1\|_E = 2F_G(A).$$

Take $a_0 \in A$ such that

$$\|a_0 - \tfrac{1}{2}(x_0 + x_1)\| = h(A, \tfrac{1}{2}(x_0 + x_1)) = \|A - \tfrac{1}{2}(x_0 + x_1)\|_E,$$

Then

$$\|a_0 - x_0 + a_0 - x_1\| = \|A - x_0 + A - x_1\|_E$$

and

$$2F_G(A) = \|a_0 - x_0 + a_0 - x_1\| \leqslant \|a_0 - x_0\| + \|a_0 - x_1\| \leqslant 2F_G(A).$$

This implies that

$$\|a_0 - x_0 + a_0 - x_1\| = \|a_0 - x_0\| + \|a_0 - x_1\|.$$

It follows from the strict convexity of X that $x_0 = x_1$, which is a contradiction. So $R_G(A)$ is a singleton.

Note that for any $A \in J(G) \cap V(G)$, the maximization problem $\max(A, G)$ has a unique solution. Now let (x_n, z_n) with $x_n \in A$, $z_n \in G$ be any maximizing sequence. Then, using Lemma 3.5 and the compactness of A, we have that (x_n, z_n) converges strongly to the unique solution and complete the proof by Lemma 3.3 and 3.6.

Remark 3.1. Theorem 3.1 is a multivalued version of results due to Asplund [1], Panda & Kapoor [22], Zhivkov [28] and Fitzpatrick [13].

Remark 3.2. Note that if $\max(A, G)$ has a unique solution (x_0, z_0) then x_0 has a unique furthest point in G, which implies that there is a dense set of X possessing unique furthest points in G provided that the result of Theorem 3.1 holds. This enables us to construct some counterexamples to which Theorem 3.1 fails if X is not strongly convex. In fact, in this case, either X is not both reflexive and strictly convex, or X is not Kadec. In the first case Example 5.3 in [13] and Remark 4.4 in [9] apply. In the second case, let X be the renormed space $l_2 \oplus R$ in [12] by taking

$$|||(x, r)||| = \max\{\|x\|, |r|\} + \left[r^2 + \sum_n 2^{-2n} x_n^2\right]^{1/2}$$

for $(x, r) \in X$. Let

$$G = \{(e_n, 2 - n^{-1}) : n = 2, 3, ...\}$$

and

$$U = \{(u, r)\colon \|u\| < \tfrac{1}{4}, |r| < \tfrac{1}{4}\}.$$

Then, for $(u, r) \in U$,

$$F_G(u, r) = 2 - r + \left[(2-r)^2 + \sum_n 2^{-2n} u_n^2 \right]^{1/2}.$$

However, for each $(e_n, 2 - n^{-1}) \in G$

$$|||(u, r) - (e_n, 2 - n^{-1})||| > F_G(u, r),$$

which shows no points in the set U has a furthest point in G. Hence Theorem 3.1 fails. Obviously, X is both reflexive and strictly convex.

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